Invariants and monovariants – lecture notes

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When faced with a system that undergoes repeated, seemingly chaotic transformations, it can be incredibly difficult to track the exact state of the system after many steps. Instead of trying to brute-force the process, we look for properties that *do not change*. This is the core idea of an **invariant**. By finding a quantity that remains constant throughout the chaos, we can bridge the gap between the initial state and the final state.

In this session, we will start by exploring invariants to tame unpredictable operations. Later on, we will look at **monovariants**—quantities that strictly move in one direction—which are excellent tools for proving that a process must eventually terminate.

1. Invariants

An easy warm-up.

Example 1. The numbers $1, 2, 3, \ldots, 2026$ are written on a blackboard. Every minute, you erase two numbers a and b and replace them with the single number $a \times b$. This process continues until only one number remains. What is that number?

Solution. Suppose our board just has the numbers $(2, 3, 4)$. If we select $a = 2$ and $b = 3$, we erase them and write $2 \times 3 = 6$. The new board is $(6, 4)$. Next, we select our only remaining pair, 6 and 4, and replace them with $6 \times 4 = 24$. The process terminates, and our final number is 24.

Notice that our initial numbers were $2, 3, 4$, and their total product was $2 \times 3 \times 4 = 24$.

From this simple example, a key observation stands out: the operation of replacing two numbers with their product does not change the *overall product* of all the numbers on the board. The total product is our **invariant**.

We can easily verify this. If the board has numbers $x_1, x_2, \ldots, x_n$, the product is $P = x_1 \times x_2 \times \cdots \times x_n$. If we replace x_1 and x_2 with $(x_1 x_2)$, the new product is $P' = (x_1 x_2) \times x_3 \times \cdots \times x_n$, which is exactly the old product.

Since the product of the numbers on the board never changes, the final single number must simply be the product of the initial numbers:

$$1 \times 2 \times 3 \times \cdots \times 2026 = 2026!$$

□

That was straightforward because the operation itself was exactly the invariant we needed. We encounter a slightly messier operation in the next example.

Example 2. The numbers $1, 2, 3, \ldots, 2026$ are written on a blackboard. Every minute, you erase two numbers a and b and replace them with the single number $a + b + ab$. This process continues until only one number remains. What is that number?

Solution. Let's test this with a small initial configuration, say $(1, 2, 3)$. If we pick $a = 1$ and $b = 2$, we replace them with $1 + 2 + (1 \times 2) = 5$. The board is now $(5, 3)$. If we then apply the operation to 5 and 3, we get $5 + 3 + (5 \times 3) = 23$. The final number is 23.

Unlike the previous problem, the sum is not constant, and the product is not constant. The expression $a + b + ab$ feels like a strange mix of addition and multiplication.

Our strategy is to find a way to manipulate this expression into something familiar. Let's look closely at the replacement rule: $a + b + ab$. This looks suspiciously close to the expansion of $(a + 1)(b + 1) = ab + a + b + 1$.

Let's check the algebra. What happens if we add 1 to our replacement rule?

$$(a + b + ab) + 1 = (a + 1)(b + 1)$$

This is a great observation! It tells us that if we were to add 1 to every number on the board, the *product* of these new shifted numbers would be our **invariant**.

Let's formally define this. Suppose the numbers currently on the board are $x_1, x_2, \ldots, x_k$. We define our invariant P as the product of each number increased by one:

$$P = (x_1 + 1)(x_2 + 1) \cdots (x_k + 1)$$

If we erase x_1 and x_2 and replace them with $y = x_1 + x_2 + x_1x_2$, let's calculate the new product P':

$$\begin{aligned} P' &= (y + 1)(x_3 + 1) \cdots (x_k + 1) \\ &= (x_1 + x_2 + x_1x_2 + 1)(x_3 + 1) \cdots (x_k + 1) \\ &= (x_1 + 1)(x_2 + 1)(x_3 + 1) \cdots (x_k + 1) \\ &= P \end{aligned}$$

The product of the numbers *plus one* is invariant!

Now we just apply this to our initial board. The initial numbers are $1, 2, 3, \ldots, 2026$. Our invariant product is:

$$P_{\text{initial}} = (1 + 1)(2 + 1)(3 + 1) \cdots (2026 + 1) = 2 \times 3 \times 4 \times \cdots \times 2027 = 2027!$$

When the process terminates, there is only one number left on the board. Let's call it X. Because our invariant must hold true, the "shifted" final product must equal our initial shifted product:

$$X + 1 = 2027!$$

Solving for X, we find that the final number is $X = 2027! - 1$. □

We can elegantly apply the idea of invariants to coloring problems as well.

Example 3. The cells of a 7×7 board are chess-painted (alternating colors) so that the corners are black. One is allowed to repaint any two adjacent cells to the opposite color. Is it possible to repaint the entire board white using such operations?

Solution. Suppose we select two adjacent cells to repaint. At any given moment, these two cells must consist of either one black and one white cell, two black cells, or two white cells. Let's see what happens to the total number of black cells on the board in each scenario:

If we choose one black and one white cell, they become one white and one black cell. The total number of black cells changes by 0. If we choose two black cells, they both become white. The total number of black cells changes by -2. If we choose two white cells, they both become black. The total number of black cells changes by $+2$.

Notice that every operation changes the total number of black cells by 0, $+2$, or -2. Therefore, we cannot change the evenness or oddness of the total number of black cells. The parity of the black cells is perfectly preserved, making it an **invariant**.

Our strategy is to use this invariant to show that the target state is unreachable. Let B be the total number of black cells on the 7×7 board. Since $\Delta B \in \{-2, 0, 2\}$, the parity of B never changes. If B starts odd, it must remain odd forever.

The board has 49 cells in total. Because the initial colors alternate and the four corners are black, there must be exactly one more black cell than white cells.

$$B_{\text{initial}} = \frac{49+1}{2} = 25$$

The initial number of black cells is 25, which is an **odd** number.

The problem asks if we can reach a state where the entire board is white. In this target state, the number of black cells would be:

$$B_{\text{target}} = 0$$

Since 0 is an **even** number, and our operation strictly preserves the parity of the black cells, it is impossible to reach the all-white state. □

Problem 1. 64 coins are arranged in an 8-by-8 square. Initially half the coins are heads-up and half the coins are tails-up, in an alternating pattern. At each step there are three possible moves: flip all eight coins in a single row; flip all eight coins in a single column; flip all four coins in a 2×2 square. Is it possible to reach a configuration where exactly one coin is heads-up and the rest are tails-up?

Problem 2 (Engel)**.** A circle is divided into six sectors. The numbers $1, 0, 1, 0, 0, 0$ are written into the sectors in counterclockwise order. At each step, you may increase two neighboring numbers by 1. Is it possible to equalize all numbers by a sequence of such steps?

Problem 3 (Tuymaada Junior 2018)**.** The numbers $1, 2, 3, \ldots, 1024$ are written on a blackboard. They are divided into pairs. Then each pair is wiped off the board and non-negative difference of its numbers is written on the board instead. 512 numbers obtained in this way are divided into pairs and so on. One number remains on the blackboard after ten such operations. Determine all its possible values.

Problem 4 (USAMO 1993)**.** Let a, b be odd positive integers. Define the sequence f_n by putting $f_1 = a$, $f_2 = b$ and letting f_n for $n \geq 3$ be the greatest odd divisor of $f_{n-1} + f_{n-2}$. Show that f_n is constant for n sufficiently large and determine the eventual value as a function of a and b.

Problem 5 (ISL 2012)**.** Several positive integers are written in a row. Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y+1, x)$ or $(x-1, x)$. Prove that she can perform only finitely many such iterations.

2. Monovariants

When you need to prove that a process eventually stops (terminates), a powerful strategy is to look for a **monovariant**.

A monovariant (also known as a semi-invariant) is a quantity associated with the state of a system that changes strictly in one direction upon each step of the process—for example, it always strictly increases or strictly decreases. Crucially, this quantity must be bounded above or below. Because an integer cannot strictly increase or decrease indefinitely while being bounded above or below respectively, any process governed by an integer monovariant must eventually terminate.

Example 4. Some positive integers are written on a board. A move consists of choosing two of the integers, x and y, and replacing them with $\mathrm{LCM}(x, y)$ and $\mathrm{HCF}(x, y)$.

Prove that eventually the numbers on the board stop changing.

Solution. Let's apply this process on several example sets. Let the initial set be $\{4, 6\}$. We select $x = 4$ and $y = 6$. Compute $\mathrm{LCM}(4, 6) = 12$ and $\mathrm{HCF}(4, 6) = 2$. The new set is $\{2, 12\}$. If we attempt to apply the operation again on 2 and 12, we obtain $\mathrm{LCM}(2, 12) = 12$ and $\mathrm{HCF}(2, 12) = 2$. The set remains unchanged; the process has terminated.

Now let the initial set be $\{4, 6, 9\}$. Select 4 and 6. Replace them with 12 and 2. The board is now $\{2, 9, 12\}$. Select 9 and 12. Then the set becomes $\{2, 3, 36\}$. Select 2 and 3. Then the board becomes $\{1, 6, 36\}$. Any further operations on pairs from this set will result in the exact same numbers, so the process has terminated.

From our examples, two things stand out:

1. Observe the products of the elements in our examples. In our first example, the initial product is $4 \times 6 = 24$, and the final product is $12 \times 2 = 24$. In our second example, the initial product is $4 \times 6 \times 9 = 216$, and the final product is $1 \times 6 \times 36 = 216$. This is simply because of a fundamental identity in number theory. For any two positive integers x and y, we have $x \cdot y = \mathrm{LCM}(x, y) \cdot \mathrm{HCF}(x, y)$. So the product of all numbers on the board remains constant, i.e. it is an **invariant**.

2. Let us examine the sum of the elements. In our first example, the sum of the numbers was $10, 14, 14, \ldots$. In our second example, the sum was $19, 23, 41, 43, 43, \ldots$. We hypothesize that the sum of the numbers on the board strictly increases with every non-trivial step (so when numbers change). It is a **monovariant**.

This is a useful observation because a sum of positive integers cannot increase indefinitely if their product remains fixed.

First, I show the sum is bounded above. Let the invariant product of all n numbers on the board be P. The maximum possible sum of n positive integers whose product is P is achieved when $n - 1$ of the numbers are as small as possible (namely, 1), and the remaining number is as large as possible (namely, P). Thus, the sum of the numbers on the board is bounded above by $P + \underbrace{1 + 1 + \cdots + 1}_{n-1 \text{ times}} = P + n - 1$.

Second, I show that the sum strictly increases with every non-trivial step.

Let the two chosen numbers be x and y, and let their greatest common divisor be $g = \text{HCF}(x, y)$. We can express x and y as $x = a \cdot g$ and $y = b \cdot g$, where a and b are coprime positive integers (i.e., $\text{HCF}(a, b) = 1$). The least common multiple is therefore $\text{LCM}(x, y) = a \cdot b \cdot g$.

The sum of the two numbers before the operation is:

$$x + y = ag + bg = g(a + b)$$

The sum of the replacement numbers is:

$$\text{LCM}(x, y) + \text{HCF}(x, y) = abg + g = g(ab + 1)$$

We must determine if the new sum is strictly greater than the initial sum. Using the fact that $g > 0$, we have:

$$\begin{aligned} & g(ab + 1) > g(a + b) \\ \iff & ab + 1 > a + b \\ \iff & ab - a - b + 1 > 0 \\ \iff & (a - 1)(b - 1) > 0 \end{aligned}$$

To justify why the final inequality holds, we must consider when a non-trivial change actually occurs on the board. If x divides y, then $\text{LCM}(x, y) = y$ and $\text{HCF}(x, y) = x$. Replacing the pair $\{x, y\}$ with $\{y, x\}$ leaves the board entirely unchanged. Therefore, a non-trivial change to the board only occurs if neither x divides y nor y divides x.

If x does not divide y, then $a > 1$. If y does not divide x, then $b > 1$. Therefore, $(a - 1) > 0$ and $(b - 1) > 0$. Their product is therefore strictly positive, which implies that the sum of the numbers on the board strictly increases.

So combining both claims we have just proven, upon every non-trivial step, the sum of the numbers on the board increases by at least 1. However, this sum is bounded above by $P + n - 1$. Because an integer sequence cannot increase indefinitely while being bounded above by a finite number, the process must eventually terminate. $\square$

Observe the final state of the second example: $\{1, 6, 36\}$. Notice 1 divides 6, and 6 divides 36.

We can explain this by looking at prime factorizations. For any prime factor, the LCM takes the highest power, while the HCF takes the lowest power. Because of this, our replacement operation is essentially just sorting these prime powers—pushing the larger powers into one number and the smaller powers into the other.

The process stops when no more sorting can happen, meaning the prime powers are completely sorted from smallest to largest across all the numbers. As a result, the final numbers on the board when sorted will always line up so that each one divides the next: $A_1 \mid A_2 \mid A_3 \mid \cdots \mid A_n$.

Example 5 (IMO 1986)**.** To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Solution. Let the five numbers be v, w, x, y, z, arranged clockwise around a circle. Let $\Sigma = v + w + x + y + z > 0$ be the sum of all five numbers.

Let's apply this process on several example sets to build intuition. Let the initial configuration be $(1, -2, 3, 4, 1)$. We select $y = -2$, with adjacent numbers $x = 1$ and $z = 3$. We replace them with $1 + (-2) = -1$, $-(-2) = 2$, and $3 + (-2) = 1$. The new board is $(-1, 2, 1, 4, 1)$. If we attempt to apply the operation again on $y = -1$, the board becomes $(0, 1, 1, 4, 1)$. All numbers are now non-negative, so the process has terminated.

Now let the initial set be $(-3, 4, -2, 5, 1)$. If we repeatedly apply the operation to the negative numbers, the sequence of states might look something like this, eventually reaching $(0, 1, 1, 1, 2)$, at which point the process terminates.

From our examples, a few key observations stand out:

1. The sum of the numbers on the board remains constant, i.e., it is an **invariant**. We can easily verify this algebraically: the new sum of the three altered vertices is $(x + y) + (-y) + (z + y) = x + y + z$, which is exactly the old sum. 2. We cannot generate larger and larger negative numbers. 3. We do not seem to get trapped in a repeating cycle of states. 4. The operation seems to "smooth" out the numbers, making them more equal over time.

Our strategy is to find a measure of "spread" that acts as our **monovariant**. Since our numbers are integers, if this quantity is bounded below (e.g., it must be non-negative), a strictly decreasing monovariant would guarantee that the process must terminate in a finite number of steps.

Without loss of generality, suppose $y < 0$. Let's first test the sum of squares as our measure of spread. Define $S = v^2 + w^2 + x^2 + y^2 + z^2$.

Let S' be the new sum of squares after applying the operation. We hope that $S' - S < 0$. Let's check the algebra:

$$\begin{aligned} S' - S &= (x + y)^2 + (-y)^2 + (z + y)^2 - (x^2 + y^2 + z^2) \\ &= (x^2 + 2xy + y^2) + y^2 + (z^2 + 2zy + y^2) - x^2 - y^2 - z^2 \\ &= 2y^2 + 2xy + 2zy \\ &= 2y(x + y + z) \end{aligned}$$

Since $y < 0$, we would need $x + y + z > 0$ for S to strictly decrease. However, there is no guarantee that $x + y + z > 0$. The sum of squares is a good idea, but we need to adapt it.

Let's investigate other quantities. Consider the sum of the squares of adjacent sums. Define:

$$U = (v + w)^2 + (w + x)^2 + (x + y)^2 + (y + z)^2 + (z + v)^2$$

Let U' be the value after the operation. Let's calculate the difference $U' - U$. Notice that the terms $(v+w)^2$ and $(z+v)^2$ are unaffected by the change to x, y, z, except that the sums involving v become $(w+x+y)^2$ and $(z+y+v)^2$.

$$\begin{aligned}
U' - U &= \left[(w+x+y)^2 - (w+x)^2\right] + \left[(x+y-y)^2 - (x+y)^2\right] \\
&\quad + \left[(-y+z+y)^2 - (y+z)^2\right] + \left[(v+z+y)^2 - (v+z)^2\right] \\
&= y(2w+2x+y) - y(2x+y) - y(2z+y) + y(2v+2z+y) \\
&= y(2w+2x+y-2x-y-2z-y+2v+2z+y) \\
&= 2y(w+v)
\end{aligned}$$

Again, $y < 0$, but we have no reason to assume $w + v > 0$.

But recall that the problem gave us a **crucial** piece of information: the total sum $\Sigma = v + w + x + y + z$ is strictly positive. Notice what happens when we combine our two attempts and consider the sum $S + U$:

$$(S' + U') - (S + U) = 2y(x+y+z) + 2y(w+v) = 2y(v+w+x+y+z) = 2y\Sigma$$

Since $y < 0$ and $\Sigma > 0$, we have $\Delta(S+U) < 0$. We have found our monovariant! Since $S + U$ is a sum of squares, it is always non-negative. An integer-valued function that strictly decreases and is bounded below must eventually stop changing, meaning the algorithm must terminate. $\square$

Alternative solution (proposed by AOPS user tc1729):

The algorithm always stops. Consider the function

$$f(x_1, x_2, x_3, x_4, x_5) = \sum_{i=1}^{5} (x_i - x_{i+2})^2$$

where indices are taken modulo 5 (so $x_6 = x_1$, $x_7 = x_2$). Clearly $f \geq 0$ always, and f is integer-valued. Suppose, without loss of generality, that $y = x_4 < 0$. By expanding the terms before and after the operation, one can show that:

$$f_{\text{new}} - f_{\text{old}} = 2\Sigma x_4 < 0$$

since the total sum $\Sigma > 0$. Thus, if the algorithm does not stop, we can find an infinite decreasing sequence of non-negative integers $f_0 > f_1 > f_2 > \cdots$. This is impossible, so the algorithm must stop. $\blacksquare$

There are in fact, other monovariants that can be used to solve this problem, for example:

$$\begin{aligned}
S = &|v| + |w| + |x| + |y| + |z| + |v+w| + |w+x| + |x+y| + |y+z| + |v+w+x| \\
&+ |w+x+y| + |x+y+z| + |y+z+w| + |z+v+w| + |v+w+x+y| + |w+x+y+z| \\
&+ |x+y+z+v| + |y+z+v+w| + |z+v+w+x|
\end{aligned}$$

which, of course, is not as elegant as the ones used in the two solutions above ;)

For difficult Olympiad problems involving iterative processes, you often need to design your own "energy" function. Finding the right **monovariant** is key to capturing the underlying structure of the moves and proving termination.

Problem 6 (Engel)**.** In the Parliament of Sikinia, each member has at most three enemies. (Assume that being an enemy is symmetric: if deputy A is an enemy of deputy B, then deputy B is an enemy of deputy A.) Prove that the parliament can be separated into two houses so that each member has at most one enemy in their own house.

Problem 7 (Dylan Yu)**.** The numbers $1, 2, \ldots, 2008$ are written on a blackboard. Every second, Jimmy erases four numbers of the form $a, b, c, a + b + c$, and replaces them with the numbers $a + b, b + c, c + a$. Prove that this can continue for at most 10 minutes.

Problem 8 (St. Petersburg 2013)**.** There are 100 numbers from the interval $(0, 1)$ on the board. Every minute we can replace two numbers a, b on the board with the roots of $x^2 - ax + b = 0$ (if it has two real roots). Prove that this process must stop at some moment.